Aspects of Proving in Group Problem-Solving
in an Undergraduate Calculus Classroom Context

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1. Introduction

As with all activities in which students in mathematics classrooms are expected to engage, the doing of proof is a fundamentally social practice. Social in the sense that proof is constructed in the interaction between and among people (e.g., Ernest, 1994), and also social in the sense that the forms and meaning of proof are themselves socio-cultural constructions (Hersh, 2008; Wilder, 1989). Much has been written about the historical development of proof, especially in Western mathematics, and, indeed, the cultural-historical roots of what constitutes rigor, evidence and conviction in the discipline of mathematics point to the deeply social aspects of mathematical practice (Hanna, 1989; Hanna, 1995; Lakatos, 1976; Wilder, 1989; see also Thurston, 1994, for a compelling analysis of mathematics as a social process from the perspective of a mathematician). This paper, however, aims to show how the doing of proof—by students in undergraduate classroom contexts—is also a fundamentally social process in which the doers are also learners who must confront and make sense of what doing proof in a classroom and for teachers means.

Proving, as a process of discovery and establishment of knowledge, is the defining practice of professional mathematics (e.g., Lakatos, 1976; Thurston, 1994). As such, proof holds a key—if not the central—position in the undergraduate mathematics curriculum.[1] Especially for

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[1] In particular in contrast to K-12 mathematics curricula. Although in recent years, K-12 curricula have sought to engage students in mathematical justification and argumentation more centrally, the role of formal proof remains minimal and problematic.

undergraduate students of mathematics, engagement in processes of proof and justification is critical for robust mathematical learning, authentic mathematical participation, and the development of meaningful mathematical identity. Students' experiences with proof in undergraduate mathematics are particularly important for students' sustained interest and study in the mathematical sciences. As concerns rise regarding the nature and quality of students' mathematics education, and in particular the declining numbers of students in the mathematical sciences at the undergraduate and graduate levels (Daempfle, 2003-2004), research relevant to students' experiences with proof and proving, how students learn to prove, and pedagogical approaches to proof—in particular at the high school and undergraduate levels—has increased (see Harel & Sowder, 2007, for a recent review). Specifically, recent research in undergraduate mathematics education has posited multiple, yet convergent, aspects of students' understandings of proof and proving that impact how students engage in and learn about mathematical proof. Much of this research, however, primarily utilizes interview or survey research methods and does not situate students' proving in authentic learning contexts, thereby failing to capture how students understand and make sense of proving in the particular social context of the classroom (Herbst & Brach, 2005).

In this paper, drawing upon methods of interaction analysis (Jordon & Henderson, 1995) and microethnography (Erickson, 2004) to analyze classroom data from an undergraduate calculus discussion section, we focus on how three aspects of students' understandings of proof and proving drawn from this existing literature and refined in our analysis—(1) students' approaches to the production of proof; (2) the epistemic status of different approaches to proof; and (3) the expectations that frame students' production of proof—dynamically shape how students make meaning of proof-type problems and construct proof. This analysis highlights

how constructs related to proof and proving that were identified in out-of-classroom research contexts emerge and are relevant when students are engaged in classroom mathematical activity in which they are held accountable as students and learners. Specifically, we illustrate how these students' proof construction involves the management of *tensions* arising out of the interplay between their expectations regarding the form their proofs should take, how they "see" and make sense of the mathematical relationships in the proof statement, and the standards to which their work is actually held accountable. We conclude that while students' beliefs and knowledge about proof and argumentation are important for explaining how students' construct proof—and therefore how they can develop increasingly more sophisticated understandings of proving—it is through this dynamic and situated interplay that we can gain insight into how school settings shape that learning process.

2. Theoretical Framework

We begin our exploration from the standpoint that proving is a social process and proofs are socio-cultural artifacts (e.g., Balacheff, 1991; Herbst & Brach, 2006; Hersh, 2008; Mariotti, 2006; see also Lave, 1988, and Saxe, 1990, for more general discussions of the social and culturally situated nature of mathematical activity). Thus, our analysis centers on key aspects of students' processes of proof and proving that illuminate not only the ways that students construct proof and the meaning of proof, but importantly how these are both related to students' interactions, expectations, and responsibilities within the undergraduate classroom context. Drawn from theoretical findings from recent research on proof and proving in undergraduate mathematics education research and further clarified in the context of our own data analysis, the theoretical framework informing this study proposes three distinct but interrelated aspects of students' constructions and understandings of proof relevant to students' proof-related activity in

classroom contexts: (1) approaches to the production of proof; (2) the status of different approaches or forms of proof in terms of intuitive understanding, determining truth, and establishing validity; and (3) the expectations that frame students' production of proof, arising from their perceptions of context and 'audience.'

2.1 Students' approaches to proof production

Recent research has described different approaches to the construction of proof that doers of mathematics (e.g., students, mathematicians) employ. Primarily drawing from analysis of interview data of students and mathematicians doing and reading proof, these studies focus on the inscriptional and conceptual forms of proof that the subjects employ and/or find understandable or convincing (Raman, 2003; Weber & Alcock, 2004, in press). These lines of research have yielded highly compatible (though not redundant) findings that, generally speaking, posit two contrasting forms of proof production, which we call *proving as symbolic manipulation* and *proving as sense-making*.

Proving as symbolic manipulation involves the construction of proof by the logical manipulation of symbolic statements in order to achieve the statement to be proved. Students will begin with a statement (usually a symbolic instantiation of the given information offered as a condition of the statement to be proved) and then act upon that statement according to algebraic procedural constraints. Raman describes a similar approach with her notion of the *procedural idea* (Raman, 2003). In her interview-based study comparing university mathematics students and mathematicians reading and construction of proofs, Raman identifies the procedural idea as an approach based on formal logical symbolic manipulations. She characterizes this kind of organizing idea for a proof as "top-down" (Raman, 2003, p. 323), meaning that the prover possesses an understanding of a procedure for a proof and proceeds from what you're "supposed

to start with" (interview data in Raman, 2003, p. 323) to what is to be proved in a manner that is unconnected to intuitive or informal understandings.

Similarly, Weber and Alcock (in press; see also 2004) describe what they call *syntactic proof production*:

> First, a student may attempt to construct this proof by working within the representation system of proof. That is, the student can choose a proof framework, list his or her assumptions, derive new assertions by applying established theorems and rules of inference, and continue until the appropriate conclusion is deduced. All this can be accomplished without considering configurations in other representation systems, such as graphs, informal arguments, or prototypical examples of relevant mathematical concepts. We call attempting to prove in this manner *syntactic reasoning* and proofs successfully produced in this way are dubbed *syntactic proof productions* (Weber & Alcock, in press, pp. 8-9).

Like Raman's notion, syntactic proof production is based upon formal manipulations and is structured according to a known proof framework (or what Raman calls a known procedure). Both also emphasize the absence of connections made to informal, intuitive or other understandings derived from alternate representations of the mathematical objects and relations.

In both lines of research, these approaches to proof production are held in contrast to ways of proving that foreground sense-making and understanding. *Proving as sense-making* emphasizes the prover's intuitive and often informal understanding of the mathematical situation as a key resource in making sense of and then also constructing proof. For Raman (2003), the *heuristic idea* captures how informal insights can provide understanding of the situation to be proved. These informal understandings may be grounded in empirical data or instantiated in different representational systems, but, in the hands of some provers, do not lead to formal proof. She contrasts this with the *key idea* (2004), a heuristic idea that a prover can and does use to construct a formal proof. The 'keyness' of this idea is that it is an informal understanding that captures a key aspect of the mathematical situation in a way that it can be formalized and productively deployed in a rigorous proof.

The key idea seems to be very similar to aspects of what Weber and Alcock (2004, in press) have identified as *semantic proof production*. While the key idea is a critical conceptual insight that allows for the translation from informal to formal proof, the semantic proof production is an approach to proof in which the prover uses what Weber and Alcock call "instantiations" of relevant mathematical objects to suggest and guide formal inferences. In other words, the semantic proof production could be understood as the process of producing formal yet meaningful proof in which the key idea is a resource.

Although these characterizations of approaches to proof production provide insight into different forms of proof and, in the case of proving as sense-making, the nature of conceptual resources that provers employ, these kinds of descriptions do not address how the processes of proving that result in these forms and draw upon these conceptual resources are accomplished in ongoing talk-in-interaction. In our analyses, we have found that proving as symbolic manipulation and proving as sense-making can each be characterized by particular inscriptional and linguistic practices. In brief, when students undertake proving as symbolic manipulation, their talk consists largely of descriptions of states of mathematical relationships and animations of actions on mathematical objects; in addition, speakers and participants (or the generalized 'you') are limited to positions as animators within conditionals hypothesizing prospective actions or directives for taking particular actions. In contrast, in moments of proving as sense-making, the participants are more often positioned as agents in processes of framing ongoing activity, and speakers animate complex coordinations of linguistic, gestural and inscriptional activity.[2]

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[2] See Ryu & Hall (2001) for an analysis of shifting participation frameworks in proof construction as shifts in linguistic and representational activity.

2.2. The epistemic status of approaches to proof

The broad literature that addresses the epistemic status of mathematical proof ranges from historical and philosophical analyses to empirical studies of epistemological beliefs about mathematics and mathematical practices. We have focused our analysis of this literature on those empirical studies that address how provers come to see or establish the truth of a mathematical statement including, in particular, how different kinds of proof can provide different senses of knowing in the prover (Fischbein, 1982; Healy & Hoyles, 2000; Raman, 2003; Recio & Godino, 2000; Weber & Alcock, 2004). In this body of work, several terms are used to describe the various kinds of knowing that a proof can produce, e.g., convincing, explanatory, ascertaining, providing conviction, providing understanding. As several of these authors point out, a primary distinction made in this research is between proofs that are explanatory versus proofs that that are convincing. Although the intended meanings of these terms are not always clear, it appears that most researchers think that proofs that explain provide an intuitive, sometimes informal, argument that makes clear *why* the statement is true, while proofs that convince establish the mathematical veracity and validity of the statement, usually in a formal manner—in other words, they establish *that* the statement is true.

Raman (2003) contrasts proofs that provide "a sense of understanding" with those that may result in "conviction", though sometimes without the attendant understanding that makes the truth of the statement meaningful (pp. 322-323). Similarly, for Weber and Alcock (2004) proof productions can be "explanatory" and/or "convincing." The primary distinction seems to lie in the meaning that the approach makes available—procedural and syntactic approaches can be logically correct, and thus convince the reader of the truth of a statement, but meaningless in terms of the mathematical objects and relationships under consideration.

In our own data, it seems clear that students can be convinced (in the dictionary sense of the term rather than the formal mathematical sense) of the truth of a mathematical statement in the absence of formal proof. This sense of conviction is perhaps closer to what Fischbein (1982) has called *internal intuitive conviction*, a person's internal knowledge of the truth of the statement grounded in intuitive conceptual understandings. He contrasts this kind of conviction with *external extrinsic conviction*, derived from formal argument and, importantly, a source of conviction external to the intuitions of the prover. Fischbein's distinction between internal-intuitive and external-extrinsic, as well as our own data analysis, suggests a distinction between what is convincing for oneself and what is convincing for others or for other purposes. Indeed, as Recio and Godino (2001), Balacheff (1991), and others have argued, ways of knowing a mathematical truth through proof are situated in institutional and social contexts and cannot be understood independent of why and for whom such arguments are produced.

2.3. Students' orientations to the expectations that frame their proving

This is particularly true of students in the contexts of schooling. For example, students often base their success on a mathematical task on their perception of the teacher's expectation for that task—how the task is completed, what the completed work looks like, etc. And thus, students may often consciously or not tailor their work to the perceived expectations of their teacher. Students' proof production and work is responsive to their understandings and perceptions of the audience they are doing mathematics for and this audience's expectations. Several researchers have described the contrasting set of expectations that students seem to attend to in the production and evaluation of proof.

Healy & Hoyles (2000) work highlights how students differently conceptualize proof when asked which arguments they might adopt versus what they expect their teachers would

grade most favorably. In particular, they found that students preferred, for themselves, arguments that they could evaluate and that they found convincing and explanatory, while they believed that their teachers would mark symbolic or algebraic arguments most highly.

Raman's (2003) discussion of private versus public argument correlates directly with the distinction previously made between providing understanding and rigorously establishing veracity. Private arguments engender a personal sense of understanding while public arguments are produced "with sufficient rigor for a particular mathematical community" (p. 320). Her contrasting interviews with university mathematicians and students demonstrate how while both groups perceived that arguments for these audiences are different, for mathematicians private and public arguments are closely linked while for students there is a disconnect between their private understandings and their expectations of formal public proof.

In another analysis in this larger project (Edwards, Farlow, Liang & Hall, 2008), we examine how students orient differently to different audiences, particularly production for oneself versus production for a public but local group versus production for a perceived, often imagined, authority. This analysis examines how, in terms of linguistic and representational activity, the ongoing construction of a group's work is interactionally designed for different recipient positions—in other words, how, when the audience shifts, the nature of the students' constructions, as achieved in talk-in-interaction, also shift.

3. Data and Methods

The study was conducted in an introductory calculus course where discussion sections were organized as "workshops" (Treisman & Fullilove, 1990) in which instructors were expected to actively support student interaction and to emphasize the need for mathematical explanation and justification. A significant portion of each section was spent with students solving problems

in small groups while the instructor moved between the groups asking for explanations and answering questions. Data were collected several weeks into the fall semester, after the student groups had settled and the format of the sections was routine. After several observations early in the semester we, together with the instructor, selected one group of students and several problems for videotaping and close analysis. The participants include the instructor (TK) and a group of three non-math major freshman undergraduate students—two female (Ariel and Amy) and one male (Josh). At the time of the study, TK was a first year graduate student in the math department, and is not a native English speaker. We chose problems that explicitly required proof or justification and that contrasted analytic/geometric content in their design. As our data collection was conducted over several weeks, the problems covered topics including limits, derivatives, and the Fundamental Theorem of Calculus. Data consist of audio/video recordings of local and public talk during the small group segments of the sections, fieldnotes of classroom observations, and post-observation video-elicited interviews conducted separately with the group of students and the instructor. Selected episodes of local and public talk from the corpus were reviewed and served as the basis for a set of semi-structured questions for the interviews.

Our study uses close interaction analysis of the audio-video records of the focus group (Jordan & Henderson, 1995), supported by a parallel analysis of the interview data. Audio-video records were content-logged and initial strips of interaction were selected for closer analysis. Using these, we developed grounded theoretical categories in an iterative fashion, then extending our analysis systematically across the data corpus (Glaser & Strauss, 1967). The iterative development of categories also included reviews of findings from the literature relating to students' beliefs and knowledge of proof and proving, thus the category scheme that frames our findings synthesizes current research findings as well as constructs grounded in our data corpus.

Our analysis of the data corpus utilizing this category scheme revealed themes across the corpus and subsequent sequential analysis of the episodes revealed how those themes were manifest (emerged and co-related) in students' ongoing problem-solving activity. We found that we could describe the emergence and relation between the themes as tensions in which the students engaged, and then analyzed the secondary interview data for confirming and disconfirming evidence of the students' feelings of tension around these issues.

4. Findings: Tensions in understanding and constructing proof

In our analysis, three central tensions emerge related to approaches to proof, the epistemic status of proof, and students' expectations due to audience: (1) the tension between "doing mathematics" in terms of symbolic machinery in order to achieve proof and making sense of the mathematical situation in order to understand the argument; (2) the tension between the adequacy and utility of symbolic and geometric approaches to proving; and (3) the tension between being personally convinced *through* intuitive understanding and producing a demonstration of proof that is perceived as adequate for convincing others. These tensions emerged across the data corpus, though to differing degrees and with different entailments. In what follows, we illustrate the tensions as they became salient in the students' work on a single problem, drawing from both our interactional/sequential analysis of the episode as well as the supporting evidence from the video-elicited interviews with the subjects.

4.1. Introduction to focal episode

In the focal episode, the three students, collectively referred to as JAA (Josh, Ariel and Amy), work on the following problem together at a chalkboard:

Given a quadratic polynomial $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers, suppose that f has two real roots, r and s. Show that $f'(r) + f'(s) = 0$.

This problem was assigned from a worksheet titled *Monotonicity and Concavity*, part of the larger unit on applications of the derivative. The course had covered definition of the derivative a few weeks earlier and had begun work on applications of the derivative two sessions earlier. Immediately prior to this episode, the students had completed and discussed with the TA their work on two related problems applying the derivative to quadratic functions (see Appendix A).

4.2. Tension between "doing" mathematics and "seeing" its meaning

As JAA begin their work on the problem, there is first tension between and then coordination of the symbolic mechanics of what the students call "doing mathematics" and "seeing the meaning" of the symbols in visual space. The tension manifests as the students move back and forth between their perceived need to manipulate symbols to arrive at a statement equivalent to $f'(r)+f'(s)=0$ and their desire to understand what the symbols refer to graphically and conceptually and how they relate to one another. Their first attempts at the problem focused on identifying symbolic relationships that they think could potentially be used to derive the proof statement ($f'(r)+f'(s)=0$). For example, they begin with Ariel's suggestion that $f'(r) = 0$ (which Josh immediately points out is not possible since "*r* is a root, its not a maximum or a minimum necessarily"), followed by another suggestion from Ariel that they begin with $f(r) = 0$ and then "plug *r* into the equation and set it equal to zero? (2) I know don't know is that gonna <u>do</u> anything?" Ariel's question points to a critical orientation of their problem-solving at this stage—they are perceiving the work of proving this statement to involve symbolic manipulations that "do" something, presumably something that yields another symbolic statement closer to the desired end state. This orientation—usually initiated by Ariel and Amy—returns several times as they make unsuccessful attempts to make headway on the problem by offering various

algebraic transformations of the proof statement as well as manipulations of the expressions for

$f'(r)$ and $f'(s)$.

However, Josh seems to want to explore what the symbols represent, and his push for

"seeing" the meaning of the symbols comes into tension with Ariel and Amy's push for symbolic

manipulation. In the excerpt below, one of the early attempts to find a productive symbolic

approach, although Josh begins by trying to algebraically transforming the proof statement by

substitution, he eventually redirects the group to consider what the symbols "represent:"[3]

Josh:	… well let's see (.) f prime of r equals two a r plus b right? *[writes f'(r)=2ar+b]* and f prime of s equals two a s plus b. *[writes f'(s)=2as+b]* well so all we have to prove I guess (3) is that these two *[points to the expressions 2ar+b and 2as+b]* are equal to each other? that would mean that r equals s=
Ariel:	=no (.) that would mean that r is equal to negative s. <u>No</u> because
Josh:	what does it what does it say exactly that we have to prove?
Ariel:	(3) one is equal to the negative of
Josh:	these that these *[looking with Amy at the worksheet and pointing to the problem]* added together=
Amy:	=have to equal zero
Josh:	equal zero (8) well what do these represent?
Ariel:	those represent (2) those represent pretty much nothing important

At the start of this excerpt, although the algebraic manipulations are incorrect, it seems that Josh

and Ariel are attempting to algebraically transform the proof statement in order to derive a

simpler relationship—i.e., what "that would mean." When this approach is unproductive, Josh

then literally and figuratively steps back and attempts to use the algebraic expressions for $f'(r)$

and $f'(s)$ to rearticulate what "we have to prove". He questioningly suggests "that these two are

equal to each other?" which he says would mean that r equals s (since the two expressions are

identical but for the substitutions of r and s). Ariel corrects him, saying that it actually would be

that r equals negative s (since the original statement is that $f'(r) + f'(s) = 0$, and hence $f'(r) = -f'(s)$). However, she immediately corrects herself, perhaps realizing that just because $f'(r) = -$

f'(s), *r* doesn't necessarily equal –*s*. The focus of their responses on the symbolic representation indicates that they are still approaching the problem as symbol manipulation, indeed, that making meaning involves following the entailments of symbolic manipulation.

However, after a significant pause during which he is looking at the worksheet and Amy and Ariel are looking at the board, Josh asks "well what do these represent?," making a bid to shift their attention from the symbolic forms to their geometric representations. Josh consistently attempts to see the mathematical relationships in geometric terms across the data corpus. In other episodes, he suggests that the group "imagine" the mathematical objects in space, and draws and gestures geometric representations of symbolic expressions as a means of "seeing what they mean." As such, it appears that when asking "what do these represent?" Josh is referring to the geometric meaning of the expressions—which he later points out is the slope of the tangent at the roots. At least for Josh, making sense of the proof statement—and hence coming to understand what it is asking them to do—involves "seeing" mathematical relationships in geometric terms.

In this particular instance, however, Josh's bid to consider what the symbols represent is unsuccessful and JAA continue attempting to deal with the proof statement as action on symbols. Specifically, Ariel rejected Josh's bid and the group returns to syntactic reasoning:

Amy: [can we] [can you plug in like]
Ariel: [so if] we [plug them] in and set it equal to zero
Josh: plug them in? Where do we plug them in?
Ariel: to our quadratic formula
Amy: can you do like, plug in
Josh: oh I see and solve for r, solve for these different things? Is that what you mean

This excerpt illustrates what we identified as the push-pull between "doing the actual mathematics of it" or the students' work of symbolic manipulation and "making sense" or developing understanding like Josh proposed to the group. In fact, this push-pull relationship is further evidenced in student interviews. For instance, Josh states:

Josh: It just strikes me, I don't know if this is what actually happened, I don't remember but it's interesting to me that there's sort of this first moment of revelation like of understanding of how it should be conceptually, but it was very clear for us that wasn't enough. We had to go on and write it down and figure out how it related back so=

Ann: Related back to?

Josh: =to what's on the board

On the one hand, Josh talks about "this first moment of revelation" when students experience understanding of the mathematical concepts, and, on the other hand, Josh talks about having to "go on and write it down" or the symbolic manipulation of doing the mathematics. Ariel points out a similar tension:

Ariel: Yeah it's kind of like you are struggling and think of what does this all relate, what do, like where do you go, you might know the answer, but you don't necessarily know like how you can use calculus to show it. There was the pause where everyone thinks and then you just it clicks, and then you have to like deal with writing it out and doing the actual mathematics of it. But at some point, it clicks and you are like, ok this is where we go.

Here Ariel states that she "might know the answer" or understand (i.e., "it clicks"), but for Ariel this is contrasted with the need these students feel to do "the actual mathematics of it."

4.3. Tension between the adequacy and utility of geometric versus symbolic argument

Alongside the tension between sense-making and symbolic manipulation, the students' actions also reveal a tension between the adequacy and utility of geometric versus symbolic argument. In particular, across the data corpus the students struggle to figure out what kinds of arguments will be seen as adequate, and while they often draw upon geometric intuitions and arguments to convince themselves, they indicate that symbolic argument is what is valued.

As the focal episode continues, the initial push-pull between symbolic manipulation and sense-making continues as further suggestions for symbolic substitutions are made (by all the students) and Josh occasionally re-attempts to link the symbolic forms to what they represent. This back-and-forth eventually leads to a coordination of symbolic elements that they have inscribed on the board (e.g., $f(r)$, $f(s)$, $f'(r)$, $f'(s)$, $-b/2a$), their geometric representations in

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inscriptional and gestural space, and their technical names (e.g., roots, derivatives, tangents,

etc.). In the excerpt that follows, Josh attempts to recast what the problem is asking them to

prove in terms he understands, and eventually makes the connection between the roots of the

quadratic and the slopes of the tangent lines at the roots in geometric space:

Josh: these are the roots (.) I mean the idea is to get the- is to find the connection between the roots:: and derivatives right? So if we show that- if this is:: if f of this:: *[points to –b/2a written on the board, which they had previously identified as the x-coordinate of the critical point]* is negative then you'll have (.) in other words we know that this is true (.) maybe that's what we need to do. We know we have two roots (.) so in that case negative b over two a is less- f of negative b over two a is less than zero. (3) right?
Ariel: why would it=
Josh: =its only its only if a is concave up
Ariel: well well what were gonna end up proving is that two a r plus b plus two a s plus b all over two is equal to negative:: (.) no never mind
Josh: but if we what were what were actually showing is that at the two roots *[uses index fingers to point to two points in space]* we have exactly opposite graphs opp opposite slopes (.) which makes sense (.) right? Because its they're they're perfectly symmetrical. That's what we wanna show (.) is that they're exactly opposite each other. *[gesturing opposite "slopes" with open hands—Figure 1]*
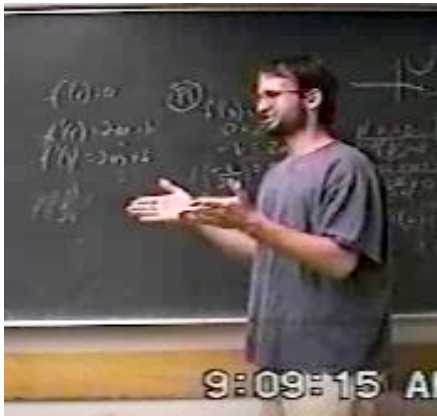

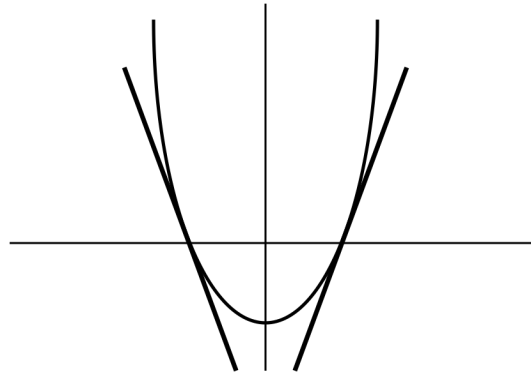Figure 1. Josh demonstrating opposite slopes of tangents at roots.

Figure 2. Rendering of Josh's drawing of parabola with two real roots and tangent lines at roots.

This excerpt begins as Josh calls attention to the roots, *r* and *s*, and suggests that "the

idea" of the proof is to "find the connection with the roots:: and derivatives right?" This move

signals a significant shift from the group's previous approach; rather than manipulating symbolic

expressions he is proposing that they consider the conceptual connections between the two of the

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key mathematical ideas in the problem, roots and derivative. This reframes the problem now as one in which symbolic expressions are used to represent images of mathematical objects and conceptual relationships, rather than simply arising as logical entailments of algebraic procedures. As such, Josh continues by describing, in verbal symbolic terms, a parabola that has two roots because it's critical point is less than zero ("We know we have two roots (.) so in that case negative b over two a is less- f of negative b over two a is less than zero. (3) right?"). Ariel also attempts to recast the proof statement, but in purely symbolic terms by substituting the appropriate expressions for the value of the derivative of f at r and s into the proof statement[4]; however, she quickly realizes that her suggestion is problematic and rejects it. Josh then offers what we have identified as the key geometric argument in the group's work: Because the graph of the quadratic function f is "perfectly symmetrical", the slopes of the tangent lines at the two roots are "exactly opposite each other." He accompanies and coordinates his verbal argument with gestural instantiations of roots (pointing at two points at the same vertical displacement in gestural space) and slopes at those roots (using his open hands to indicate the angles of opposite slopes at the imagined points in gestural space—Figure 1). After a significant 11-second pause, Josh then moves to the board and slowly and deliberately draws a graphical rendering of his argument (Figure 2)[5]:

> Josh: (11) If they're <u>opposite</u> each other *[drawing a parabola]*
> (7) so that *[drawing tangent at left root]*
> (4) and that *[drawing tangent at right root]*
> (38) *[All looking at drawing and looking away]*

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[4]This substitution was, according to the TA, one of the expected approaches to the proof, and several other groups in the section did complete the proof this way. Ariel's attempt is incorrect, however, as she seems to confuse the substitution into the proof statement with a previous attempt using substitution. In any case, her symbolic substitution attempt is not taken up by her groupmates.
[5] The particularity of Josh's drawing—a parabola whose line of symmetry is the y-axis and that is concave up—raises interesting issues about deriving generality of argument from particular inscribed geometric cases. See Brown, 1997, for a related discussion of generalization and pictures as proofs.

Thus, the push-pull process between sense-making using geometric representation and "doing mathematics" as symbolic manipulation culminates in Josh's production of an argument, performed in a complicated coordination of verbal, gestural and inscriptional action, that demonstrates that the slopes of the tangents at the two roots of a concave up parabola (with vertex below the x-axis) are opposite. However, he frames this argument not as a "proof" that concludes their work on the problem, but as a clarification of "what we're actually showing", a recasting of what the problem is asking them to prove in ways that "make sense" to them. Indeed the group's orientation away from geometric argument as an adequate production and toward symbolic forms is dramatically evidence in what follows the production of the drawing. After Josh's demonstration of his geometric argument, the group stands looking at the drawing and looking away in silence for 38 seconds. Although it is impossible to determine with certainty what they were thinking during this period of silence, it is very clear that they are not acting as if they are done with the proof. Work is left to be done, and although they realize that Josh has provided a geometric argument, the students seem to perceive a lingering need for a symbolic argument. When asked about this period of silence in interview, Josh and Ariel explain:

Josh: Like after those 35 seconds, there's a big loss of hope. Whenever I'm thinking about something there's also the idea that while I'm thinking the other people are also thinking that might give us the right direction. As it keeps going and going and going and nobody says anything and nobody has any ideas and we're just kinda plugging and plugging and plugging, I start to get this feeling like oh well where is he? He'll get here soon, the answer will be here soon.

Ariel: Especially when you understand the concept. For me, I know what I'm trying to do, I know it's just algebra, where's the TA.

Here it is clear that the students "understand the concept" and are satisfied that Josh's geometric argument provided understanding, but they are aware that they still need to do the "algebra" or produce a symbolic argument. So in fact we see that while geometric argument is useful (to them) for understanding the problem, it is inadequate for doing proofs in this context. And while

symbolic argument is what counts, it is "notational nonsense," as Josh calls it in the context of another problem, or "just algebra" that just needs to get done.

4.4. Tension between convincing oneself versus producing proof for others

The 38 seconds of silence mark the transition from the group's making sense of "what we're really showing" and what they orient to as the actual work of proof production. The group then begins a series of attempts at producing chains of symbolic statements ending with the proof statement. As in their earlier attempts, they fail to make productive manipulations, and this time no connections to the underlying ideas of the symbols are explored. The emphasis of their work is instead on identifying logically appropriate symbolic substitutions and following the logical entailments of particular features of symbolic statements (e.g., if there is '$+c$' on both sides, they cancel out; if multiple terms contain the same symbol, it should be factored out; etc.). The understanding drawn from Josh's geometrically-based demonstration is not taken up as a "guide" for making formal logical inferences nor served as a "key idea" in the development of their formal deductive argument (Raman, 2003; Weber & Alcock, 2004).[6]

At this point TK—the instructor—comes over to the group and asks them to explain their work. In particular, he has noticed Josh's drawing (Figure 2) and, as he explains in interview, and he is probing for their understanding of the key insight—symmetry—that the figure shows. The group presents Josh's geometric argument, explaining that they all understood it, to which TK tells the group that this argument is acceptable and to move on to the next problem. In surprise, the students question TK:

Ariel: but is that all you want us to do?
Josh: but what kind of steps do you want us do you want us to prove it I mean

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[6] As several colleagues have pointed out, the key geometric insight in this particular problem is non-trivial to transform into a formal symbolic proof. That these students do not successfully accomplish this should not be surprising; perhaps more noteworthy is that they do not attempt to do so.

TK: uhh
Josh: theoretically we could prove the whole thing //geometrically]
TK: //yeah] you could. So uh I'm more satisfied if you if you just see it (.) right I
         mean (??) The most important steps would be that the that this parabola is (.)
Josh: symmetrical
TK: symmetric symmetrical
Josh: but I saw that even in my drawing

This illustrates the tension that arose for students between what they found convincing for

themselves and the expectations they perceive that frame what they think it takes to convince

others. Indeed, these students found Josh's geometric argument personally convincing, but they

did not think that this argument counted as something that was convincing for others. In a later

interview Ariel reveals that this tension caused feelings of frustration for her:

Ariel: It's frustrating. I always think in terms of the test, I like there to be a standard,
         this is the norm this is how far you have to take it. I would prefer it to be how it
         was where you just understand the concept and that's fine, but I know that
         sometimes that's not what they want, they want more, but they're not that specific
         about it. So it's just frustrating, how far do you want me to go? Do want me to
         understand it, or do you want me to be able to prove to anyone who hasn't a clue
         about any of this?

On the one hand Ariel wants to know if she expected to do what is convincing for her personally

or if she is expected to produce arguments that are convincing for "anyone who hasn't a clue

about any of this."

5. The duality of learner vs. student: Tensions in learning to prove

As we look across the tensions discussed above we see that students manage two roles—

their role as students and their role as learners. As students, they perceive an expectation in the

classroom to convince others using a syntactic approach to proof production. And as learners,

they have personal desires for understanding and personal conviction arrived at using geometric

argumentation. Furthermore, we see in this analysis that these tensions manifest through

students' feeling of frustration and uncertainty.

This raises issues regarding how students orient to the production of proof; in this classroom context, as is likely the case for many traditional school contexts, proof is not to be produced for the purpose of convincing oneself, it is produced in order to convince others. But to convince others of what? It is clear from Ariel's statement about her frustration that her concern is not about what it takes to be convincing to others (whoever they are), but about how to convince those in authority (the TA, the people who grade her test) *that she can produce adequate proof*. Adequacy, however, is not something she can come to understand or make sense of, it is a social standard or norm that she is held to and to which she must become acculturated. Thus, for Ariel and her groupmates, and likely many undergraduate calculus students, learning to prove is not a matter of learning how to convince, but of divining the standards and norms of rigor that their instructors expect but often leave unarticulated.[7]

6. Conclusions/Implications

This study, situated in an authentic Calculus classroom context and focused on the dynamics of processes of proving as they unfold, provides practical contributions to the teaching and learning of mathematical justification and proof and theoretical contributions to existing research on students' conceptions of and approaches to proof. Clearly, there are implications in this work for Calculus instructors, teaching assistants, and/or course developers whose responsibilities are based around students' successful learning. An understanding of the possible tensions that students manage while developing proofs may help to inform teachers of the nature of the challenges and choices that students face. For example, an awareness of the multiple

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[7] Others have also noted the lack of authenticity of the vast majority of proof-related mathematical activities in classrooms; in contrast to the purposes of proving in the disciplinary practices of professional mathematics, students in most learning contexts prove something that is already known (why else would they be asked to prove it) as a vehicle for engaging in particular content (e.g., derivative) and/or explicitly exercising proof production (e.g., in a transitions to advanced mathematics undergraduate course). They are not engaging in mathematical discovery or establishment of truth or validity.

audiences that JAA had in mind while trying to produce a proof may have prepared TK for the students' reactions of uncertainty when he accepted their geometric proof. Furthermore, with such an awareness, he could have taken advantage of the opportunity for explicit discussion of the function and purposes of proof for students. More broadly, these tensions raise the issue of the sometimes contradictory messages about proof that undergraduate students (and presumably others) receive with regard to making sense of and producing convincing and rigorous arguments. It should be considered whether they reflect larger tensions in the mathematics educational community about argument versus proof, understanding versus production, and the relative values different representational demonstrations of mathematical argument that are themselves reflected in the wider arenas of mathematical practice and the history and philosophy of mathematics.

Theoretically, our discussion of tensions within a Calculus classroom focused on proof is a contribution to the current theoretical understanding of students' group work on proof-based problems. Situating this study in an authentic classroom context highlights how students' frame their work by the understandings of the norms and expectations to which the are held accountable, and reveals the specific representations and forms of argument they employ as they attempt to meet those norms and standards. This analysis also sheds light on how the inauthenticity of students' proof activity in undergraduate calculus courses not only shapes how they produce proof, but also calls into question what about doing proof is being learned in those situations. Research must acknowledge the fundamentally social nature of mathematical activity, and of proof in particular, and, thereby, in the context of examining students' learning to prove, pay closer attention to how students are attending to and being socialized into proving practices (especially those practices that are at odds with what mathematicians and math educators

consider authentic and productive). Finally, from a broader perspective, as our findings underscore the relationships between aspects of students' understandings and beliefs about proving as they emerge in interaction situated in particular contexts, they provide a window onto the interaction between cognition and social context.

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Appendices

Appendix A: Focal problem

In the focal episode, the students were working on Problem 1c below. They had just completed Problems 1a and 1b.

1. a. Show that a quadratic polynomial $f(x)=ax^2 + bx + c$, where a, b, and c are real numbers, always has one critical point and no points of inflection. When is f concave up? When is f concave down?
   b. How can you tell if a quadratic polynomial has two roots? One root? No roots?
   c. Suppose that f has two real roots, r and s. Show that $f'(r) + f'(s) = 0$.

Appendix B: Transcription Conventions

, (.)	short pause
(#)	pause for # seconds
.	falling tone
?	rising tone
::	extended syllable
-	self-interrupt
=	latching; no pause between utterances
underline	emphasis/stress
//]	overlapping talk
[ ]	overlapping talk and gesture
(??)	not intelligible
( )	text in parentheses is unclear
((LF))	laughing
{italics}	gestures